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Cellular networks as models for Planck-scale physics

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Abstract. Starting from the working hypothesis that both physics and the corresponding mathematics have to be described by means of discrete concepts on the Planck scale, one of the many problems one has to face in this enterprise is to find the discrete protoforms of the building blocks of our ordinary continuum physics and mathematics. We base our own approach on what we call '*cellular networks*', consisting of cells (nodes) interacting with each other via bonds (figuring as elementary interactions) according to a certain '*local law*'. Geometrically our dynamical networks are living on *graphs*. Hence a substantial amount of the investigation is devoted to the development of various versions of discrete (functional) analysis and geometry on such (almost random) webs. Another important topic we address is a suitable concept of *intrinsic (fractal) dimension* on erratic structures of this kind. In the course of the investigation we make comments concerning both different and related approaches to quantum gravity as, say, the *spin network framework*. It may perhaps be said that certain parts of our programme seem to be a realization of ideas sketched by Smolin some time ago.

1. Introduction

There exists a suspicion in parts of the scientific community that nature may be discrete or rather 'behaves discretely' on the Planck scale. But even if one is willing to agree with this 'working philosophy', it is far from evident what this vague metaphor might actually mean or how it can be implemented into a concrete and systematic inquiry concerning physics and mathematics in the Planck regime.

There are basically two attitudes to 'discreteness on the Planck scale'. One comprises approaches which start (to a greater or lesser degree) from continuum concepts (or more specifically, concepts more or less openly inspired by them) and then try to detect or create modes of 'discrete behaviour' on very fine scales, typically by imposing quantum theory in full or in part upon the model system or framework under discussion. There are prominent and very promising candidates in this class like, for example, *'string theory'* or *'loop quantum gravity'*. Somewhat intermediate is a more recent version (or rather, aspect) of the latter approach, its *'polymer'* respectively *'spin network'* variants. As these approaches are widely known we refrain from citing the vast corresponding literature. We recommend instead [1, 2] as recent reviews of the latter approach, containing some cursory remarks about the former together with a host of references, and, as a beautiful introduction to the conceptual problems of quantum gravity in general [3].

Alternatively one can adopt an even more speculative and radical attitude and approach the Planck regime from the opposite direction by developing a framework with 'discreteness' already built in and then try to reconstruct, 'bottom up' (so to speak), all the continuum concepts of ordinary spacetime physics as *'collective quantities'* such as, for example

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'collective excitations' via the cooperation of many microscopic (discrete) degrees of freedom. If one were very bold one could even entertain the idea that the quantum phenomena are perhaps not the eternal and irreducible principles they are still viewed as today by the majority of physicists but, rather, may come to be seen as derived and secondary concepts, together with gravitation, emerging from a more primordial, truly discrete and *'combinatorial'* theory. As for corresponding strategies, see the perhaps prophetic remarks of Penrose [4] and the ideas presented in [5, p 267]. In a wider context see also the ideas of Sorkin and Balachandran *et al* [6], while the approach of 't Hooft [7] is noteworthy because of its emphasis on the possible usefulness of *cellular automata* (i.e. simpler and more rigid variants of our *cellular networks*).

We would like to add a clarifying remark concerning [5]. The mentioned ideas are sketched near the end of a long contribution and we stumbled upon them only recently (February 1998). One could perhaps say that the framework we develop here (and in related papers [8–11] is, at least in part, an implementation of Smolin's programme. One of the main motivations of both Smolin's and our approach is the hope that, proceeding in this way, the peculiar form of nonlocality or entanglement observed in quantum theory can actually be implemented and thus better understood.

It goes without saying that such a radical approach is beyond the reach of direct experimental verification in the strict sense for the foreseeable future (as with the other frameworks mentioned above). Rather one has to rely on inner theoretical criteria: among other things, the capability to generate the hierarchy of increasingly complex patterns we are observing in nature or in present day *'effective theories'*, which describe the various regimes many orders of magnitude away from the Planck scale, while introducing as few, simple and elementary assumptions as possible. One would like such a framework to provide clues as to how the continuum concepts of ordinary spacetime physics/mathematics may emerge in a natural manner from their respective *'discrete protoforms'*.

Another more aesthetic criterion would be a kind of natural convergence of the different approaches towards a common substructure which is discrete in a really primordial way. Indications for such a convergence can be detected in various lines of research going on presently. *'Spin networks'* and *'polymer states'* are cases in point where modes of discreteness emerge from an, at first glance, continuous environment. It may well be that *'string field theory'* will turn out to live on a more discrete and singular substratum than presently suspected (some speculative remarks pointing in this direction can be found at the end of [12]), a catchword being *'fractal geometry'*. A brief but illuminating analysis concerning such a possible convergence in the future towards a decidedly discrete and *'combinatorial'* common limit is given in section 8 of [1]. A group of related ideas with which we sympathize is developed by Nottale [13].

In this paper we embark on the development of a conceptual framework which has a pronounced combinatorial flavour and makes contact with various branches of modern discrete mathematics such as *algebraic combinatorics*, *(random) graph theory* and *noncommutative geometry*. Our argument is in three parts. Section 2 deals with the dynamics of '*cellular networks*'. Our approach is different from perhaps more canonical approaches which rely in a more or less open way on ideas already employed in the continuum versions such as '*fields*', '*path integrals*' or a relative regular substratum like a periodic lattice structure. In contrast, we regard most of these concepts as '*emergent structures*' according to our working philosophy. In other words, these patterns need not have identifiable counterparts on the primordial scale. Thus we share the philosophy of Smolin, Ashtekar, and quite a few others (see for example, section 6.1 in [14]), that even if it is agreed that a continuum theory has to be understood as the coarse-grained limit of an underlying

'*critical*' discrete and more fundamental theory, that is no sufficient reason to surmise that the primordial building blocks or the dynamics at the fundamental core of our universe will share a lot of similarities with the (almost) continuum concepts of our '*effective theories*' living in a regime of at most intermediate energies many orders of magnitude away from the Planck regime.

In section 3 we show how one can introduce versions of *discrete analysis* and *geometry* into erratic environments. At the same time we compare our approach with other existing ones (being, in part, derivatives of noncommutative geometry in the sense of Connes *et al*).

Finally, in section 4 we develop dimensional concepts on such discrete spaces. We would like to stress that we consider it of crucial importance not to base such concepts on anything which could be identified as a variant of a continuum or embedding space. As to these ideas much more can be found in [8], in particular concerning various conceptual relations to *fractal geometry*.

We close this introduction with a clarifying remark. To discretize continuum quantum gravity (or rather certain models), e.g. to put it on a lattice or triangulate it, is of course not new, but most of the approaches we are aware of discretize continuum theories living in a fixed embedding space with a definite dimension in the standard sense. Furthermore the kind of discretizations being employed are typically relatively regular while in our approach the underlying graphs are rather random structures with vertices and bonds as dynamical variables which can even be switched on or off. As a consequence the dimension of our networks is an '*emergent*' and dynamical, perhaps even *fractal*, property of the system. To mention only two representative papers of the many others employing or reviewing strategies of the former type see [15, 16] and references therein. The thesis of Bakker [17] and the lecture of Lee [18] should perhaps also be mentioned in this context. This does not of course mean that there are no relations between these different approaches but we prefer to discuss them elsewhere in order to keep this paper readable.

2. The cellular network environment

In this section we sketch the type of model systems on which the following analysis will be based. As already mentioned, we start from a rather primordial level, trying to make no allusions whatsoever to continuum concepts. We then show how protoforms of ideas and notions that play a key role in ordinary continuum physics/mathematics emerge in a relatively natural and unforced way from this framework. Cases in point are concepts such as *'dimension'*, *'differential structure'*, the idea of *'physical points'* (being endowed with an internal structure), the web of which establishes the substratum of macroscopic spacetime, and other geometrical/topological notions. The framework even turns out to be rich enough to support a fully fledged kind of *'discrete functional analysis'*, comprising *'Laplace–Dirac operators'* etc. It is noteworthy that an advanced structure like the *'Connes' spectral triple'* shows up very naturally in this context.

Besides the reconstruction of basic concepts of continuum physics and mathematics another goal is to describe the microdynamics going on in this discrete substratum over (in) which macroscopic spacetime is floating as a kind of coarse-grained *'superstructure'*. The formation of *'physical points'* and their mutual entanglement, yield the kind of *'near-/far-order'* or *'causal structure'* we are used to from continuum spacetime.

To this end we view this substratum as, what we like to call, a *'cellular network'*, consisting of *'nodes'* and *'bonds'*. The nodes are assumed to represent certain elementary modules (cells or 'monads') having a discrete, usually simple, internal state structure, the bonds modelling elementary direct interactions among the nodes. As an important

ingredient, these bonds are dynamical in so far as they can be in a (typically limited) number of *'bond states'*, thus implementing the varying strength of the mutual interactions among the cells.

It is a further characteristic of our model class that these interactions are not only allowed to vary in strength but, can be switched off or on, depending on the state of their local environment. In other words, stated in physical terms, bonds can be created or annihilated in the course of network evolution, which (hopefully) enables the system to undergo *'geometric phase transitions'* accompanied by an *'unfolding'* and *'pattern formation'*, starting from a less structured chaotic initial phase. To put it briefly: in contrast to, say, *'cellular automata'*, which are relatively rigid and regular in their wiring and geometric structure (especially with the bonds typically being nondynamical), our cellular networks do not carry such a rigid overall order as an external constraint (e.g. a regular lattice structure); their 'wiring' is dynamical and thus behaves randomly to some extent. The key is that order and modes of regularity are expected to emerge via a process of *'self-organization'*.

Definition 2.1 (class of cellular networks).

(1) 'Geometrically' our networks represent at each fixed *'clock time'* a *'labelled graph'*, i.e. they consist of nodes $\{n_i\}$ and bonds $\{b_{ik}\}$ or $\{d_{ik}\}$ (see the next section), with the bond b_{ik} connecting the nodes (cells) n_i, n_k. We assume that the graph has neither elementary loops nor multibonds, that is, only nodes with $i \neq k$ are connected by at most one bond.

(2) At each site n_i we have a local node state $s_i \in q \cdot \mathbb{Z}$ with q, for the time being, a certain not further specified elementary quantum. The bond variables J_{ik}, attached to b_{ik}, are in the simplest cases assumed to be two- or three-valued, i.e. $J_{ik} \in \{\pm 1\}$ or $J_{ik} \in \{\pm 1, 0\}$.

Remark.

(1) In the proper graph context the notions *'vertex'* and *'edge'* are perhaps more common (see e.g. [19]). As for some further concepts used in graph theory see below.

(2) This is, in some sense, the simplest choice one can make. It is an easy matter to employ instead more complicated internal state spaces like, say, groups, manifolds etc. One could in particular replace $\mathbb{Z}$ by one of its subgroups or impose suitable boundary conditions.

(3) In the following section we will give the bonds b_{ik} an *'orientation'*, i.e. (understood in a precise algebraic/geometric sense) $b_{ik} = -b_{ki}$. This implies the compatibility conditions $J_{ik} = -J_{ki}$.

Next we have to impose a dynamical law on our model network. In doing this we are of course inspired by *'cellular automaton laws'* (see e.g. [20]). The main difference, however, is that in our context the bonds are dynamical degrees of freedom and *a fortiori* can become dead or alive (active or inactive), so that the whole net is capable of performing drastic topological/geometrical changes in the course of clock time.

A particular type of a dynamical '*local law*' is now introduced. We assume that all the nodes/bonds at '*(clock) time*' $t+\tau$, (τ an elementary clock time-step), are updated according to a certain local rule which relates for each given node n_i and bond b_{ik} their respective states at time $t + \tau$ to the states of the nodes/bonds of a certain fixed local neighbourhood at time t.

It is important that, generically, such a law does not lead to a reversible time evolution, i.e. that there will typically exist attractors in total phase space (the overall configuration space of the node and bond states).

A crucial ingredient of our network laws is what we would like to call a '*hysteresis interval*'. We assume that our network, called in the following QX ('*quantum space*'), starts

from a densely entangled '*initial phase*' QX_0, in which practically every pair of nodes is on average connected by an '*active*' bond, i.e. $J_{ik} = \pm 1$. Our dynamical law will have a built-in mechanism which switches bonds off (more properly: sets $J_{ik} = 0$) if local fluctuations among the node states become too large. There is then hope that this mechanism may trigger an '*unfolding phase transition*', starting from a local seed of spontaneous large fluctuations towards a new phase (an attractor) carrying a certain '*superstructure*', which we would like to relate to the hidden discrete substratum of spacetime (points).

One example of such a law is given in the following definition.

Definition 2.2 (Local law). At each clock time-step a certain '*quantum*' q is transported between, say, the nodes n_i, n_k such that

$$s_i(t+\tau) - s_i(t) = q \cdot \sum_k J_{ki}(t) \tag{1}$$

(i.e. if $J_{ki} = +1$ a quantum q flows from n_k to n_i etc).

The second part of the law describes the back reaction on the bonds (and is, typically, more subtle). This is the place where the so-called '*hysteresis interval*' enters the stage. We assume the existence of two '*critical parameters*' $0 \leqslant \lambda_1 \leqslant \lambda_2$ with.

$$J_{ik}(t+\tau) = 0 \qquad \text{if } |s_i(t) - s_k(t)| =: |s_{ik}(t)| > \lambda_2 \tag{2}$$

$$J_{ik}(t+\tau) = \pm 1 \qquad \text{if } 0 < \pm s_{ik}(t) < \lambda_1 \tag{3}$$

with the special proviso that

$$J_{ik}(t+\tau) = J_{ik}(t) \qquad \text{if } s_{ik}(t) = 0. \tag{4}$$

On the other hand

$$J_{ik}(t+\tau) = \begin{cases} \pm 1 & J_{ik}(t) \neq 0 \\ 0 & J_{ik}(t) = 0 \end{cases} \qquad \text{if } \lambda_1 \leqslant \pm s_{ik}(t) \leqslant \lambda_2. \tag{5}$$

In other words, bonds are switched off if local spatial charge fluctuations are too large, switched on again if they are too small, (their orientation following the sign of local charge differences) or remain inactive.

Remark.

(1) The reason we do not choose the 'current' $q \cdot J_{ik}$ proportional to the 'voltage difference' $(s_i - s_k)$ as e.g. in Ohm's law is that we favour a *nonlinear* network which is capable of *self-excitation* and *self-organization* rather than *self-regulation* around a relatively uninteresting equilibrium state! The balance between dissipation and amplification of spontaneous fluctuations has, however, to be carefully chosen ('*complexity at the edge of chaos*').

(2) We have emulated these local network laws on a computer. It is not yet clear whether this simple network law does everything we expect. Nonetheless, it is fascinating to observe the enormous capability of such intelligent networks to find attractors very rapidly, given the enormous accessible phase space.

(3) In the above class of laws a direct bond–bond interaction is not yet implemented. We are prepared to incorporate such a contribution step if it turns out to be necessary, although there are not so many ways to do this in a sensible way. Stated differently, the class of possible physically sensible interactions is perhaps not so numerous.

(4) Note that, in contrast to, for example, Euclidean lattice field theory, the *'clock time'* t is, for the time being, not standing on the same footing as potential 'coordinates' in the network (e.g. curves of nodes/bonds). We rather suppose that so-called *'physical time'* will

emerge as a sort of secondary collective variable in the network, i.e. will be different from the clock time (while being of course functionally related to it).

In our view remark 4 is consistent with the spirit of relativity. What Einstein was really teaching us is that there is a (dynamical) interdependence between what we experience as space, respectively time, not that they are absolutely identical! In any case, the assumption of an overall clock time is only made for convenience in order to not make the model system too complicated. If our understanding of the complex behaviour of the network dynamics increases, this assumption may be weakened in favour of a possibly local and/or dynamical clock frequency. A similar attitude should be adopted concerning concepts like *'Lorentz-(in)covariance'* which we also consider as *'emergent'* properties. Needless to say it is of tantamount importance to understand how these patterns emerge from the relatively chaotic background, a question that will be addressed in future work.

As can be seen from the definition of the cellular network, a full-scale investigation of its behaviour separates quite naturally into two parts of both a different mathematical and physical nature. One part comprises its more geometric/algebraic content in the form of large static graphs and their intricate structure (at, say, arbitrary but fixed clock time), thus neglecting the details of the internal states of bonds and nodes, the other conveys a more dynamical flavour, analysing and keeping track of the topological/geometrical change and pattern formation in the course of clock time. Both parts represent an intricately entangled bundle of complicated problems and require the development or application of a fair amount of quite advanced (discrete) mathematics. We concentrate here on the former part. As for the latter part, an impression of the impending problems is given in [11] which, however, should only be considered a preliminary draft.

Before we embark on studying in detail the '*pregeometric*' patterns in cellular networks we want to briefly comment on the more general structure of evolution laws in such systems. The above is only one candidate from a whole class of such laws. For one thing, it is quite evident that the '*local state spaces*' living over the respective nodes and bonds can be chosen in a more general way. For another, the local dynamical law can also be chosen to be more general.

Definition 2.3 (general local law on cellular networks). Each node n_i can be in a number of internal states $s_i \in \mathcal{S}$. Each bond b_{ik} carries a corresponding bond state $J_{ik} \in \mathcal{J}$. Then the following general transition law is assumed to hold:

$$s_i(t+\tau) = ll_s(\{s_k(t)\}, \{J_{kl}(t)\}) \tag{6}$$

$$J_{ik}(t+\tau) = ll_J(\{s_l(t)\}, \{J_{lm}(t)\}) \tag{7}$$

$$(\underline{S}, \underline{J})(t+\tau) = LL((\underline{S}, \underline{J})(t)) \tag{8}$$

where ll_s, ll_J are two maps (being the same over the whole graph) from the state space of a local neighbourhood of the node or bond on the l.h.s. to $\mathcal{S}$, $\mathcal{J}$, yielding the updated values of s_i and J_{ik}. $\underline{S}$ and $\underline{J}$ denote the global states of the nodes and bonds and LL the global law built from the local laws at each node or bond.

Irrespective of the technical details of the dynamical evolution law under discussion it should emulate the following, in our view crucial, principles, in order to match certain fundamental requirements concerning the capability of '*emergent*' and '*complex*' behaviour.

(1) As is the case with, say, gauge theory or general relativity, our evolution law on the primordial level should encode the mutual interaction of two fundamental substructures. Put sloppily: '*geometry*' acting on '*matter*' and vice versa, where in our context '*geometry*' is assumed to correspond in a loose sense with the local and/or global bond states and

'*matter*' with the structure of the node states. (We will not comment further on this working philosophy here as it represents the basis of forthcoming work.)

(2) By the same token the above self-referential dynamical circuit of mutual interactions is expected to favour a kind of '*undulating behaviour*' or '*self-excitation*' above a return to some uninteresting '*equilibrium state*' which is frequently found among systems consisting of a single component which directly acts back on itself. This propensity for the '*autonomous*' generation of undulation patterns is in our view an essential prerequisite for some form of '*protoquantum behaviour*' we hope to recover on some coarse-grained and less primordial level of the network dynamics.

(3) In the same sense we expect the possibility of switching on and off of bonds to generate a kind of '*protogravity*'.

We close this section with a short aside concerning the definition of evolution laws of '*spin networks*' by Markopoulou and Smolin and Borissov ([21, 22]). As in our case there are two possibilities: treating evolution laws within an integrated spacetime formalism or regarding the network as representing space alone with the time evolution being implanted via some extra principle (which is the way we have chosen above). Closely connected with this question is the development of a suitable concept of *dimension* in this context. We start to develop our own concept in the final section of this paper. More information can be found in [8]. As the interrelation of these various approaches and frameworks is both very interesting and presently far from obvious we plan to compare them elsewhere.

3. Discrete analysis on graphs and networks

In the following we will show that despite their discreteness graphs and networks are capable of supporting a surprisingly rich differential and geometric structure, so that the catchword '*discrete analysis*' in the heading of this section is no exaggeration. One can even develop a fully fledged '*discrete functional analysis*' with Hilbert spaces, Laplacian graph, Dirac operator etc. This then leads into the fascinating field of describing geometric structures and patterns on the graph with the help of functional analytic tools (see [10]).

While our original approach has been developed from a somewhat different perspective nevertheless in the course of evolution of our framework there emerged various regions of contact and overlap with other approaches such as, for example, what is called '*noncommutative geometry*' (see e.g. [23, 24]. As a brief but concise introduction we also recommend [25]). If a certain part of this highly abstract machinery is applied to, say, discrete sets, one gets a version of discrete calculus which is still a relatively abstract scheme as long as it is not interpreted within a concrete model theory. Such an abstract calculus has been developed by Dimakis and Mueller-Hoissen *et al* [26]. In the following we will attempt, among other things, to relate these in some respects different, in other respects related approaches to our own framework.

We develop below various schemes with only the first related in some respects to noncommutative geometry. The others stand on a more or less independent footing. In this first approach we treat the network as a static *labelled graph*, consisting solely of *nodes* and *bonds*. We start with some theoretical graph concepts.

Definition 3.1 (simple locally finite (un)directed graph).

(1) We write the 'simple' graph as $G := (V, E)$ where V is the countable set of nodes $\{n_i\}$ (or vertices) and E the set of bonds (edges). The graph is called simple if there do not exist elementary 'loops' and 'multiple edges'. In other words, each existing bond connects two different nodes and there exists at most one bond between two nodes. (We could of

course also discuss more general graphs.) Furthermore, for simplicity, we assume the graph to be connected, i.e. two arbitrary nodes can be connected by a sequence of consecutive bonds called an 'edge sequence' or 'walk'. A minimal edge sequence, that is one with each intermediate node occurring only once, is called a 'path' (note that these definitions may change from author to author).

(2) We assume the graph to be '*locally finite*', that is, each node is incident with only a finite number of bonds. Sometimes it is useful to make the stronger assumption that this '*vertex degree*', v_i (number of bonds being incident with n_i), is globally bounded away from ∞.

(3) One can give the edges both an '*orientation*' and a '*direction*' (these two slightly different geometric concepts are frequently intermixed in the literature). Here we adopt the following convention. If two nodes n_i, n_k are connected by a bond, we interpret this to mean that there exists a '*directed bond*', d_{ik}, pointing from n_i to n_k and a *directed bond*, d_{ki}, pointing in the opposite direction. In an algebraic sense, which will become clear below, we call their '*superposition*'

$$b_{ik} := d_{ik} - d_{ki} = -b_{ki} \tag{9}$$

the corresponding '*oriented bond*' (for obvious reasons, the directions are fixed while the orientation can change its sign). In a sense the above reflects the equivalence of an '*undirected graph*' with a '*directed multigraph*' having two directed bonds pointing in opposite directions for each undirected bond.

We now take the elementary building blocks $\{n_i\}$ and $\{d_{ik}\}$ as basis elements of a certain hierarchy of vector spaces over, say, $\mathbb{C}$ with scalar product

$$\langle n_i | n_k \rangle = \delta_{ik} \qquad \langle d_{ik} | d_{lm} \rangle = \delta_{il} \cdot \delta_{km}. \tag{10}$$

Definition 3.2 (vertex-, edge-space). The vector spaces (or *modules*) C_0 and C_1, consist of the finite sums

$$f := \sum f_i n_i \qquad \text{and} \qquad g := \sum g_{ik} d_{ik} \tag{11}$$

f_i, g_{ik} ranging over a certain given *field* like e.g. $\mathbb{C}$ or ring such as e.g. $\mathbb{Z}$ in the case of a module.

Remark.

(1) These spaces can be easily completed to *Hilbert spaces* (as in [10]) by assuming

$$\sum |f_i|^2 < \infty \qquad \sum |g_{ik}|^2 < \infty \tag{12}$$

if one chooses the field $\mathbb{C}$.

(2) One can continue this row of vector spaces in a way which is common practice in, say, *algebraic topology* (for more details see below). In this context they could equally well be called '*chain complexes*'.

(3) Evidently the vector spaces could also be viewed as *discrete function spaces* over the *node, bond set* with n_i, d_{ik} representing the elementary *indicator functions*.

We now introduce two linear maps between C_0, C_1 called for obvious reasons '*boundary-*' and '*coboundary map*'. On the basis elements they act as follows.

Definition/observation 3.3 ((co)boundary operator).

$$\delta : d_{ik} \to n_k \qquad \text{hence} \qquad b_{ik} \to n_k - n_i \tag{13}$$

$$d : \; n_i \to \sum_k (d_{ki} - d_{ik}) = \sum_k b_{ki} \tag{14}$$

and linearly extended. That is, δ maps the directed bonds d_{ik} onto the terminal node and b_{ik} onto its 'boundary', while d maps the node n_i onto the sum of the 'ingoing' directed bonds minus the sum of the 'outgoing' directed bonds or on the sum of 'oriented' ingoing bonds b_{ki}.

The following results show that these definitions lead in fact to a kind of '*discrete differential calculus*'.

Observation 3.4 (discrete differential forms). From the above it follows that

$$\mathrm{d}f = \mathrm{d}\Big(\sum f_i n_i\Big) = \sum_{k,i}(f_k - f_i)d_{ik}. \tag{15}$$

One could now enter the field of true *discrete functional analysis* by defining the so-called '*Laplacian graph*'.

Definition/observation 3.5 (Laplacian graph).

$$\delta d f = -\sum_i \Big(\sum_k f_k - v_i \cdot f_i\Big)n_i = -\sum_i \Big(\sum_k (f_k - f_i)\Big)n_i =: -\Delta f \tag{16}$$

where v_i denotes the node degree or 'valency' defined above and the k-sum extends over the nodes adjacent to n_i.

For more results along these lines see [10]. However, we prefer to develop a framework which is more in the spirit of '*discrete geometry*' or abstract '*combinatorial topology*'.

3.1. The graded semi-differential algebra of strings or walks

The elementary algebraic/geometric building blocks of our framework are the nodes and (un)directed bonds. Next we use them to form '*edge sequences (walks)*' or '*strings*'.

Definition/observation 3.6 (admissible strings). With C_0, C_1 denoting the vector spaces spanned by the nodes and bonds, C_k comprises finite sums of admissible edge sequences consisting of k consecutive edges or $(k+1)$ consecutive nodes. We write them as

$$d_{i_0 i_1} \cdot d_{i_1 i_2} \ldots d_{i_{k-1} i_k} \qquad \text{or} \qquad n_{i_0} \ldots n_{i_k} \tag{17}$$

where by admissible we mean that each pair of consecutive nodes is connected by a bond. The multiplication sign standing between the respective edges is explained below and the strings are now interpreted as normalized basis elements in the vector space C_k over, say, $\mathbb{C}$.

Remark. Note that the repeated occurrence of a particular node or bond in such a string is not forbidden, whereas two consecutive nodes are always different. *A fortiori* a substring of the form $d_{ik} \cdot d_{ki}$ or $n_i n_k n_i$ is admissible. Such strings or substrings are algebraically useful if one wants to have the notion of '*inverse string*' and a corresponding multiplication structure (thinking of more general algebraic concepts such as groupoids etc).

We can now extend the (co)boundary maps δ, d to consecutive pairs of spaces, C_k, C_{k+1} by combining geometric imagery with experience from algebraic topology.

Definition/observation 3.7. With a certain admissible string $n_0 \ldots n_k$ given (for notational simplicity we write k instead of i_k) we define δ, d as follows:

$$\delta(n_o \ldots n_k) := \sum_{i=0}^{k}(-1)^i n_0 \ldots \widehat{n_i} \ldots n_k \tag{18}$$

where the hat means that the respective node has to be deleted with the proviso that the string is mapped to zero if n_{i-1}, n_{i+1} are not connected by a bond, i.e. if the new string is not admissible.

The extension of the coboundary operator runs as follows:

$$d(n_0 \dots n_k) := \sum_{n_\nu, i} (-1)^i n_0 \dots n_{i-1} n_\nu n_i \dots n_k \tag{19}$$

where i runs from 0 to $(k+1)$ and admissible nodes are inserted before the node n_i or for $i = (k+1)$ after the last node n_k. The insertion runs over all nodes being connected with both n_{i-1} and n_i.

The geometric picture behind this scheme is quite transparent. A given string in C_k is mapped onto a superposition of strings in C_{k-1} or C_{k+1} with appropriate weights as prefactors.

After these preparatory steps we can now set up a certain discrete differential structure with a strong geometric flavour. We have the $\mathbb{N}_0$-graded vector space

$$C := \sum C_k \qquad k \in \mathbb{N} \cup \{0\} \tag{20}$$

with the '*raising*' and '*lowering*' operators d and δ, mapping the '*level sets*' of homogeneous elements, C_k, into each other.

In a first step we make C into a '*graded algebra*'.

Definition/observation 3.8 (graded algebra). We define multiplication of homogeneous elements via '*concatenation*', i.e.

$$(n_{i_0} \dots n_{i_k}) \cdot (n_{j_0} \dots n_{j_l}) = (n_{i_0} \dots (n_{i_k} \cdot n_{j_0}) \dots n_{j_l}) \in C_{k+l} \tag{21}$$

where $n_{i_k} \cdot n_{j_0}$ means the natural algebra product structure on C_0, i.e. pointwise multiplication

$$f \cdot g = \left(\sum f_i n_i \right) \cdot \left(\sum g_i n_i \right) = \sum f_i g_i n_i \tag{22}$$

that is $n_{i_k} \cdot n_{j_0} = \delta_{i_k j_0} n_{j_0}$.

Geometrically this describes the glueing together of strings or walks, the product being zero if the end node of the first string is different from the initial node of the second string (in other words, this algebra has a lot of '*zero divisors*').

As a consequence C has a natural '*bimodule structure*' over C_0.

Corollary 3.9 (bimodule structure). With

$$n_i \cdot (n_i \dots n_k) = (n_i \cdot n_1) n_2 \dots n_k = \delta_{i1} \cdot n_1 \dots n_k \tag{23}$$

$$(n_1 \dots n_k) \cdot n_i = n_1 \dots (n_k \cdot n_i) = \delta_{ki} \cdot (n_1 \dots n_k) \tag{24}$$

and linear extension C becomes a bimodule over the algebra C_0. Algebraically n_1, n_k play the role of '*left-*' and '*right-*' identities with respect to the above string.

In order to exhibit the connection to differential calculus in the style of Connes or Dimakis *et al*, we make the following observation concerning the interplay of algebraic and differential structure.

Observation 3.10. Given an admissible string $n_0 \dots n_k$ and applying the operator d on each of the nodes $n_1, \dots, n_k$ we have the identities

$$n_i n_{i+1} = n_i \cdot d n_{i+1} = d n_i \cdot n_{i+1} \tag{25}$$

and hence

$$n_0 \dots n_k = (n_0 n_1) \cdot (n_1 n_2) \dots (n_{k-1} n_k) \tag{26}$$

(concatenation of bonds, i.e. elementary strings)

$$= (n_0 dn_1) \cdot (n_1 dn_2) \ldots (n_{k-1} dn_k) = n_0 \cdot dn_1 \cdot dn_2 \ldots dn_k =: n_0 dn_1 \ldots dn_k. \tag{27}$$

Proof. The latter manipulations exploit the algebraic (concatenation) rules we have already established above (remember the definition of dn_i as a certain superposition of 1-strings). The former identities hold because

$$n_i \cdot d_{i+1} = n_i \cdot \left(\sum_k (d_{k(i+1)} - d_{(i+1)k}) \right) \tag{28}$$

with $d_{k(i+1)}$ denoting the string $n_k n_{i+1}$ etc, hence $n_i \cdot d_{k(i+1)} = \delta_{ik} \cdot d_{i(i+1)}$. That is, it survives only the one term $d_{i(i+1)}$ or $n_i n_{i+1}$ since the other term $n_i \cdot d_{(i+1)k}$ is always zero by the definition of the concatenation product. □

It is exactly these abstract (usually uninterpreted) objects $a_0 da_1 \ldots da_k$ which occur in the abstract formalism of, say, '*noncommutative de Rham complexes*' or the '*universal differential algebra*' (see e.g. [23] or for the case of discrete sets [26]).

Remark.

(1) In an earlier version [27] we based most of the algebraic calculus on the concept and properties of tensor products (such as modules) which may have obscured somewhat the very suggestive geometric imagery. On the other hand it is perhaps closer to the abstract framework of noncommutative geometry.

(2) In the same paper we showed that our strings also realize the abstract structure of a '*groupoid*'. For the sake of brevity we omit the development of the corresponding framework in this paper.

The above shows that our strings form a graded algebra with concatenation as multiplication. In a next step we show that the coboundary operator d fulfills the so-called (graded) '*Leibniz rule*' (well known from e.g. the exterior differential algebra)

Observation 3.11 (graded Leibniz rule). With w_1, w_2 two strings, it holds

$$d(w_1 \cdot w_2) = dw_1 \cdot w_2 + (-1)^{\deg(w_1)} w_1 \cdot dw_2 \tag{29}$$

where $\deg(w_1) =$ number of edges in w_1

In the special case of zero-strings or functions from C_0 the analogous formula reads

$$d(f \cdot g) = df \cdot g + f \cdot dg \tag{30}$$

as $\deg(f) = 0$.

Proof. Whereas the latter result is contained in the former one, we also supply an additional proof.

$$d(f \cdot g) = \sum_i f_i g_i dn_i = \sum_i \left(\sum_k (n_k n_i - n_i n_k) \right) \tag{31}$$

$$df \cdot g + f \cdot dg = \left(\sum_{i,k} f_i g_i (n_k n_i) - \sum_{i,k} f_i g_k (n_i n_k) \right) + \left(\sum_{i,k} f_k g_i (n_k n_i) - \sum_{i,k} f_i g_i (n_i n_k) \right). \tag{32}$$

The mixed $f_i g_k$-terms in the latter equation cancel each other (after a change of dummy variables), hence the result.

In the case of general strings with

$$w_1 = n_{i_0} \dots n_{i_k},\ w_2 = n_{j_0} \dots n_{j_l} \tag{33}$$

we can split the action of d on $w_1 \cdot w_2$ into two groups of terms, one comprising the action *within* w_1 or w_2, the other consisting of the terms which arise from the action of d on the '*interface*' between w_1 and w_2.

The terms in the first group occur also in $dw_1 \cdot w_2$ if d acts within w_1 or in $w_1 \cdot dw_2$ modified by an extra weight factor $(-1)^{\deg(w_1)}$ if d acts within w_2.

As to the interface terms, there are three possibilities.

(1) n_{i_k} and n_{j_0} are not nearest neighbours, i.e. they are more than one bond distance apart. In that case $w_1 \cdot w_2 = 0$, hence $d(w_1 \cdot w_2) = 0$. On the other side, under this proviso no term in dw_1 is incident with w_2 and vice versa, that is $dw_1 \cdot w_2 = 0 = w_1 \cdot dw_2$.

(2) $n_{i_k} \neq n_{j_0}$ but they are nearest neighbours. This implies $w_1 \cdot w_2 = 0 = d(w_1 \cdot w_2)$. Now there is exactly one term in dw_1 incident with w_2, namely $n_{i_0} \dots n_{i_k} n_{j_0}$ carrying the weight $(-1)^{i_k+1}$. A corresponding term occurs in dw_2, carrying the weight $(-1)^0 = 1$. In the graded Leibniz rule this latter term (standing in front of $w_1 \cdot dw_2$) is multiplied by $(-1)^{i_k}$, yielding the superposition

$$(-1)^{i_k}((-1)^1 \cdot (\textit{string}) + (-1)^0 \cdot (\textit{string})) = 0. \tag{34}$$

(3) The remaining possibility, $n_{i_k} = n_{j_0}$ is very simple. The corresponding end term in dw_1 leads away from $n_{i_k} = n_{j_0}$. So its concatenation with w_2 is zero. The same holds for the initial term in dw_2. That is, in this situation all the nonzero terms in $dw_1 \cdot w_2$, $w_1 \cdot dw_2$ are internal terms, falling in the first group, discussed above, and for which the stated identity has already been verified.

This proves the graded Leibniz rule for strings (a different proof is given in [28]). □

We have now to explain why we call our algebra a '*semidifferential algebra*'. To this end we show that in general, i.e. if the underlying graph is not '*complete*' or a '*simplex*', the relations $d \cdot d = 0$ and $\delta \cdot \delta = 0$ do not hold for every string or node. On the other hand they are fulfilled for a complete graph. Phenomena like these are analysed in more detail (among other things) by Nowotny in his diploma thesis ([28]) and are not difficult to prove in full generality. Therefore we content ourselves with giving the main line of the reasoning (the phenomenon was already discussed in [27, section 4.3]).

Take, e.g. a graph, consisting of the nodes n_0, n_1, n_2 and the bonds n_0n_1, n_1n_2 (plus the 'opposite bonds' n_1n_0, n_2n_1). We have

$$d(n_0) = -n_0n_1 + n_1n_0 \qquad \text{and} \qquad dd(n_0) = -n_0n_1n_2 + n_2n_1n_0 \neq 0. \tag{35}$$

On the other hand, if the bonds n_0n_2, n_2n_0 were present we would have

$$dn_0 = -n_0n_1 - n_0n_2 + n_2n_0 + n_1n_0 \tag{36}$$

and it is easy to see that all the terms in ddn_0 do in fact cancel.

Analysing the general case a little more systematically we observe the following. In the '*reduced graph*' (i.e. some bonds missing) the following can happen. Apply d to a given string which yields, e.g. an insertion between node n_{i-1} and node n_i, e.g. $\dots n_{i-1}n_\nu n_i \dots$ with a weight $(-1)^i$. Applying d again may yield another admissible insertion of the type $\dots n_{i-1}n_\mu n_\nu n_i \dots$ coming with the weight $(-1)^i \cdot (-1)^i$ provided that n_μ is connected with n_i and n_ν.

On the other hand the 'counterterm' with the weight $(-1)^i \cdot (-1)^{i+1}$ may be missing as it can happen that n_μ is connected with n_{i-1} and n_ν but *not* with n_{i-1} and n_i so that $\ldots n_{i-1} n_\mu n_i \ldots$ does not show up in the first step in contrast to the analogous term with n_ν. A similar result holds for δ, hence we have the following.

Conclusion 3.12. We call the above algebra a 'semidifferential algebra' since in general (if some bonds are missing) $dd \neq 0, \delta\delta \neq 0$ on certain strings. If, on the other hand, the underlying graph is complete, we have $dd = \delta\delta = 0$.

These last observations are perhaps of some help if one wants to compare our 'bottom up' approach with a more 'top down' approach, which starts from the concept of the '*universal differential algebra*' (see the remarks at the beginning of this section). It is then an almost trivial observation (corresponding results being almost ubiquituous in abstract algebra) that every '*differential algebra*' is the homomorphic image of the universal one. Put differently, it is isomorphic to the universal algebra divided by a certain '*(differential) ideal*' (see below). These results have then been applied to discrete sets by e.g. Dimakis and Mueller-Hoissen *et al* [26].

We also want to stress that models, in our case networks or graphs, tend to have a more interesting geometric or algebraic structure than suggested by such an abstract viewpoint and should be analysed for their own sake (see also the following sections). We now briefly comment on one particular feature that one should be aware of if one favours the abstract approach. We call it the '*problem of unnatural relations*'.

To conform with the more abstract notation adopted in the above-mentioned literature (or [27]), we now denote the universal differential algebra over a given set of nodes (i.e. the complete graph) by $\Omega^u = \sum \Omega^u_k$, the reduced algebra (i.e. the actually given graph with some bonds missing in general) by $\Omega = \sum \Omega_k$. Furthermore, d_u is the differential on Ω^u with $d_u \cdot d_u = 0$. Note that in Ω^u_k each sequence of $(k+1)$ nodes is an admissible string.

We can now define a projector Π which projects Ω^u onto Ω by

$$\Pi(n_0 \ldots n_k) = 0 \tag{37}$$

if $(n_0 \ldots n_k)$ is not admissible in Ω

$$\Pi(n_0 \ldots n_k) = n_0 \ldots n_k \tag{38}$$

if $(n_0 \ldots n_k)$ is admissible.

Consequence 3.13. We have

$$\Pi = \Pi^2 \qquad \Omega^u = \Pi\Omega^u + (\mathbb{I} - \Pi)\Omega^u \tag{39}$$

with $\Pi\Omega^u = \Omega$. We could now continue by defining

$$d := \Pi \circ d_u \circ \Pi \tag{40}$$

which leaves Ω invariant but in general $d \cdot d \neq 0$ in contrast to $d_u \cdot d_u = 0$ as we have seen above.

We observe that $\mathrm{Ker}\,(\Pi)$ is a two-sided ideal I consisting of the elements $n_{0\ldots k}$ having at least one pair of consecutive nodes *not* being connected by a bond in the reduced graph. However, this ideal I is *not* left invariant under the action of d_u! A closer analysis shows that $d_u(n_{0\ldots k}) \notin I$ if d_u creates '*insertions*' between nonconnected neighbours in the reduced graph such that nonadmissible elements become admissible, i.e. connected. We hence have the following.

Observation 3.14. In general there exist elements $n_{0\ldots k} \in \mathrm{Ker}\,(\Pi)$ such that $\Pi(d_u(n_{0\ldots k})) \neq 0$, in other words, I is in general not left invariant by d_u.

If one wants to make Ω into a real differential algebra one has to enlarge I!

Consequence 3.15. The ideal $I' = I + d_u \circ I$ is invariant under d_u and d defines a differential algebra on the smaller algebra $\Omega^u/I' \subset \Omega$ with $\Omega = \Omega^u/\mathrm{Ker}(\Pi)$.

(That I' is an ideal left invariant by d_u is easy to prove with the help of the property $d_u \cdot d_u = 0$.)

So far so good, but a certain problem now emerges. Ω^u/I' is the algebra one automatically arrives at if one defines the homomorphism Φ from Ω^u to the reduced differential algebra in the following canonical way:

$$\Phi : n_i \to n_i \qquad d_u n_i \to d n_i \tag{41}$$

i.e. under the premise that d defines already another differential algebra. It is in this restricted sense that the above-mentioned general result has to be understood.

However, this may (and in general, will) lead to a host of unnatural relations in concrete models, such as e.g. our network: models which may already carry a certain physically motivated interpretation going beyond being a mere example of an abstract differential algebra. Note e.g. that in our algebra Ω an element like n_{123} is admissible (i.e. nonzero) if n_1, n_2 and n_2, n_3 are connected. n_{123} may, however, arise from a differentiation process (i.e. from an insertion) like $d_u(n_{13})$ with n_1, n_3 *not* connected.

This is exactly the situation discussed above.

$$n_{13} \in I \qquad \text{but} \qquad d_u(n_{13}) \notin I. \tag{42}$$

Dividing by I' maps $d_u(n_{13})$ onto zero whereas there may be little physical or geometric reason for n_{123} or a certain combination of such admissible elements being zero in our network.

Conclusion 3.16. Given a concrete physical network Ω one has basically two choices. Either one makes it into a fully fledged differential algebra by imposing further relations which may, however, be unnatural from a physical point of view and very cumbersome for complicated networks. This was the strategy followed in [26]. Or alternatively one considers Ω as the fundamental object and each admissible element in it as being nonzero. As a consequence the corresponding algebraic/differential structure on Ω may be less smooth at first glance ($dd \neq 0$ in general), although considerably more natural! At the moment we refrain from making a general judgement about the respective advantages of these two different approaches although we incline towards the latter one.

3.2. The graph as a (higher) simplicial complex

In the previous section the geometric building blocks have been strings or walks, that is, more or less one-dimensional objects. Now we turn our attention to higher dimensional patterns. We start from a fixed graph G with vertex set V and edge set E. In the literature graphs are frequently considered as '*one-dimensional simplicial complexes*' with the nodes as 0-simplices and the bonds as 1-simplices. We think this point of view is unnecessarily restrictive (while there exist of course certain mathematical reasons for this restriction we do not mention). Having our own working philosophy of networks as models for microscopic spacetime in mind, we think this concept should be generalized.

Definition 3.17 (subsimplices). We call a subset of $(k+1)$ vertices a 'k-simplex', s_k, or 'complete subgraph' if every two nodes of s_k are connected by a bond.

Observation 3.18. If the node degree, v_i, of n_i is smaller than ∞ then n_i can lie in at most a finite set of different simplices, an upper bound being provided by the number of different subsets of bonds emerging from n_i.

Proof. Assume that s_k, s_l are two different simplices containing n_i. By definition n_i is linked with all the other nodes in s_k or s_l. As these sets are different by assumption, the corresponding subsets of bonds emerging from n_i are different. On the other hand, not every subset of such bonds corresponds to a simplex (there respective endpoints need not form a simplex), which proves the upper bound stated above. □

Consequence 3.19.

(1) The set of subsimplices is evidently 'partially ordered' by inclusion.

(2) Furthermore, if s is a simplex, each of its subsets is again a simplex (called a 'face').

(3) It follows from observation 3.18 that each of these 'chains' of linearly ordered simplices is finite with the same upper bound as in observation 3.18. In other words each chain has a maximal element, a so-called 'maximal subsimplex' (MSS). By the same token each node lies in at least one (but generically several) MSS.

(4) Such a MSS with n_i being a member can comprise at most $(v_i + 1)$ nodes. In other words, its cardinality is the minimum of these numbers when n_i varies over the MSS.

Remark. Such MSS's are in combinatorics or graph theory called '*cliques*'. Their potential physical role as building blocks of the microstructure of spacetime in our particular approach has been discussed in greater depth in [11].

Observation 3.20. The class of simplices, in particular the MSS, containing a certain fixed node, n_i, can be generated in a completely algorithmic way, starting from n_i. The first level consists of the bonds with n_i an end node, the second level comprises the triples of nodes ('triangles'), $(n_i n_k n_l)$, with the nodes linked with each other and so forth. Each level set can be constructed from the preceding one and the process stops when a MSS is reached.

Remark. Note that at each intermediate step, i.e. having already constructed a certain particular subsimplex, there are several possible ways to proceed. However, a chain of such choices may differ at certain places from one place to another but still lead to the same simplex (being simply a permutation of the nodes of the latter simplex) in the end.

Next we give the simplices an '*orientation*'.

Definition 3.21 (orientation). The above simplices can be oriented via

$$n_{\pi(0)} \dots n_{\pi(k)} = \operatorname{sgn}(\pi) \cdot n_0 \dots n_k \tag{43}$$

π being a permutation of $(0 \dots k)$.

This definition can be used to give the set of simplices an algebraic structure by identifying the simplices which fall in the same equivalence class with respect to $\operatorname{sgn}\pi$. The respective equivalence class will henceforth be denoted by $(n_{i_0} \dots n_{i_k})$ and we make these classes into an orthonormal basis of some vector space of, say, finite sums over e.g. $\mathbb{C}$. The other class (negative orientation) is then simply the opposite vector $-(n_{i_0} \dots n_{i_k})$.

Consequence 3.22. The above observations show that our set of oriented simplices establish what is called in algebraic topology an 'abstract' or 'combinatorial' 'simplicial complex'. If the node degree is uniformly bounded away from ∞ on V, this simplicial complex has a 'finite degree' (which is the cardinality of the largest occurring MSS).

Remark. By means of an elegant argument one can show that such a simplicial complex with degree, say, n can always be embedded in $\mathbb{R}^{2n+1}$ (see e.g. [30]).

On this simplicial complex we can now define two operators δ, d which, however, differ from those defined in section 3.1. We will show in particular that $\delta \cdot \delta = d \cdot d = 0$ always holds.

Definition/observation 3.23. The 'boundary operator' δ acts in the following way on the basis elements:

$$\delta(n_0 \dots n_k) := \sum_{l=0}^{k} (-1)^l \cdot (n_0 \dots \widehat{n_l} \dots n_k) \tag{44}$$

with $\widehat{n_l}$ being omitted as usual. Note that the terms on the r.h.s. are again simplices (faces). It is a standard procedure to prove that $\delta \cdot \delta = 0$ holds. The 'coboundary operator' d is defined via

$$d(n_0 \dots n_k) := \sum_{n_\nu} (n_\nu n_0 \dots n_k) \tag{45}$$

the sum on the r.h.s. running over the nodes n_ν so that the terms under the sum form again $(k+1)$-simplices (or, put differently, the sum extends over all nodes with terms set to zero if they do not form $(k+1)$-simplices). It holds $d \cdot d = 0$.

Remark. Note that in the above formula it is important to work with equivalence classes. Otherwise the definition would become quite cumbersome. As in the case of '*exterior algebra*' of e.g. *forms* the ordering in $(\dots)$ is '*anticommutative*', that is,

$$(n_\nu n_0 \dots n_k) = -(n_0 n_\nu \dots n_k). \tag{46}$$

Proof. We show that $d \cdot d = 0$ in fact holds. We have

$$dd(s) = \sum_{\mu} \sum_{\nu} (n_\mu n_\nu s) \tag{47}$$

with s a certain simplex. As both $(n_\nu n_\mu s)$ and $(n_\mu n_\nu s) = -(n_\nu n_\mu s)$ occur in the double sum all the terms on the r.h.s. cancel. □

Observation 3.24. If we consider the s_k as basis elements of the vector space introduced above, it follows that d is the adjoint of δ.

Proof.

$$\langle s_k | \delta s'_{k+1} \rangle = \sum_{\nu=0}^{k+1} (-1)^\nu \cdot \langle s_k | (n'_0 \dots \widehat{n'_\nu} \dots n'_{k+1}) \rangle. \tag{48}$$

We see that at most one nonzero term can occur on the r.h.s., say, $(-1)^{\nu}\langle s_k| \pm s_k\rangle$, $\pm$ depending on the respective orientation. On the other hand

$$\langle ds_k|s'_{k+1}>\rangle = \sum_{n_\nu}\langle(n_\nu n_0 \dots n_k)|(n'_0 \dots n'_{k+1})\rangle. \tag{49}$$

Again there exists at most one nonzero term on the r.h.s. with the node sequence on the l.h.s. of the scalar product being a certain permutation of the node sequence on the r.h.s. As a first step we permute the $n_0 \dots n_k$ on the l.h.s. so that they are standing in the same relative order as on the r.h.s. This yields the same prefactor ± 1 as in the formula above. We then commute the additional node n_ν until it occupies the same position as in s'_{k+1} yielding another prefactor $(-1)^{\nu}$. □

Remark. This latter version is perhaps closer to the original point of view adopted in *cohomology theory* (in the sense of, say, Alexander), where cotheory typically lives on '*dual spaces*' (cf e.g. [29]).

Corollary 3.25. The above notion of orientation shows that e.g. the 1-simplices (i.e. bonds) correspond rather to the oriented bonds $b_{ik} = n_i n_k - n_k n_i$ than to the directed bonds $d_{ik} = n_i n_k$ introduced in section 3.1 (see e.g. item 3 of definition 3.1). The definition of d, given in this section, can then be rewritten as

$$d(n_0) = \sum_{n_k}(n_k n_0) = \sum_k b_{k0} \tag{50}$$

and hence turns out to be consistent with the previous one given in section 3.1, at least on the first level.

On the higher levels the correspondence is slightly trickier: see the following remark.

Remark. For higher simplices the correspondence is slightly more involved. Take e.g. a *triangle* $(n_0 n_1 n_2)$. In the spirit of section 3.1 one can take all the possible 3-strings which can be built over this simplex, in other words, all the possible permutations of $n_0 n_1 n_2$ and define the algebraic, oriented simplex $(n_0 n_1 n_2)$ of section 3.2 as follows

$$(n_0 n_1 n_2) := \sum_{per} \mathrm{sgn}(per) \cdot n_{i_0} n_{i_1} n_{i_2} \tag{51}$$

with the sum running over all the permutations of $(0, 1, 2)$ and the terms on the r.h.s. being *strings* in the sense of section 3.1. It follows directly that the sum is anticommutative with respect to rearrangements of the nodes (it may be convenient to add a global combinatorial factor $(k+1)!^{-1}$ on the r.h.s.). From this point of view the correspondence is reminiscent of the way exterior forms are built from tensors.

3.3. Yet another version of discrete calculus

Here we briefly comment on yet another version of discrete calculus, which is, at first glance, perhaps the most natural one and which is absent in approaches starting from the abstract universal differential algebra. As this variant is developed further in the diploma thesis of Novotny (some results can already be found in [11]), in particular concerning the method of '*discrete Euler–Lagrange variation*', we will not elaborate much on this point.

Definition 3.26 (partial forward derivative at node (i)).

$$\nabla_{ik} f(i) := f(k) - f(i) \tag{52}$$

where n_i, n_k are 'nearest-neighbour-nodes', i.e. connected by a bond.

Observation 3.27.

$$\begin{aligned}\nabla_{ik}(f \cdot g)(i) &= (f \cdot g)(k) - (f \cdot g)(i) \\ &= \nabla_{ik} f(i) \cdot g(i) + f(k) \cdot \nabla_{ik} g(i) \qquad (53) \\ &= \nabla_{ik} f(i) g(i) + f(i) \nabla_{ik} g(i) + \nabla_{ik} f(i) \nabla_{ik} g(i). \qquad (54)\end{aligned}$$

In other words the 'derivation' ∇ does *not* obey the ordinary Leibniz rule. In fact, application of ∇ to, say, higher powers of f becomes increasingly cumbersome (nevertheless there is a certain systematic in it). One gets for example (with $q := \nabla_{ik}$):

$$\begin{aligned}q(f_1 \dots f_n) &= \sum_i f_1 \dots q(f_i) \dots f_n \\ &\quad + \sum_{ij} f_1 \dots q(f_i) \dots q(f_j) \dots f_n + \dots + q(f_1) \dots q(f_n). \qquad (55)\end{aligned}$$

Due to the discreteness of the formalism and, as a consequence, the inevitable bilocality of the derivative there is no chance to get something like a true Leibniz rule on this level.

It is perhaps worth mentioning that the above establishes an interesting abstract algebraic multiplication structure which is called in [25] a '*Cuntz algebra structure*' (occurring there, however, in another context).

Observation 3.28 (Cuntz algebra).

$$q(f \cdot g) = q(f) \cdot g + f \cdot q(g) + q(f) \cdot q(g) \tag{56}$$

and analogously for vector fields $\sum a_{ik} \nabla_{ik}$.

With $u := 1 + q$ we furthermore get

$$u(f \cdot g) = u(f) \cdot u(g) \tag{57}$$

and

$$q(f \cdot g) = q(f) \cdot g + u(f) \cdot q(g) \tag{58}$$

i.e. a 'twisted derivation' with u an endomorphism on some algebra, $\mathcal{A}$, of functions on V.

As is the case with ∇_{ik}, the product rule for higher products can be inferred inductively:

$$\begin{aligned}q(f_1 \dots f_n) &= \sum_i f_1 \dots q(f_i) \dots f_n \\ &\quad + \sum_{ij} f_1 \dots q(f_i) \dots q(f_j) \dots f_n + \dots + q(f_1) \dots q(f_n). \qquad (59)\end{aligned}$$

Definition 3.29 ((co)tangential space). We call the space spanned by the ∇_{ik} at node n_i the tangential space T_i. Correspondingly we introduce the space spanned by the d_{ik} at node n_i and call it the cotangential space T_i^* with the d_{ik} acting as linear forms over T_i via

$$\langle d_{ik} | \nabla_{ij} \rangle = \delta_{kj}. \tag{60}$$

Definition/observation 3.30. Higher tensor products of differential forms at a node n_i can now be defined as multilinear forms

$$\langle d_{ik_1} \otimes \dots \otimes d_{ik_n} | (\nabla_{il_1}, \dots, \nabla_{il_n}) \rangle := \delta_{k_1 l_1} \times \dots \times \delta_{k_n l_n} \tag{61}$$

and linear extension.

Next we extend these concepts to functions $f \in C_0$ and *'differential operators'* or *'vector fields'* $\sum a_{ik}\nabla_{ik}$. We have to check whether this is a natural(!) definition.

Observation 3.31. Vector fields $v := \sum a_{ik}\nabla_{ik}$ are assumed to act on functions $f = \sum f_i n_i$ in the following manner

$$v(f) := \sum a_{ik}(f_k - f_i)n_i \tag{62}$$

i.e. they map $C_0 \to C_0$.

Corollary 3.32. Note that this implies

$$\nabla_{ik}n_k = n_i \qquad \nabla_{ik}n_i = -n_i \tag{63}$$

$$\nabla_{ki}n_k = -n_k \qquad \nabla_{ki}n_i = n_k. \tag{64}$$

Observation 3.33. 'Differential forms' $\omega = \sum g_{ik}d_{ik}$ act on vector fields $v = \sum a_{ik}\nabla_{ik}$ according to

$$\langle\omega|v\rangle = \sum g_{ik}a_{ik}n_i. \tag{65}$$

With these definitions we can calculate $\langle df|v\rangle$ with

$$df = \sum (f_k - f_i)d_{ik}. \tag{66}$$

Hence

$$\langle df|v\rangle = \left\langle \sum (f_k - f_i)d_{ik}| \sum a_{ik}\nabla_{ik} \right\rangle = \sum (f_k - f_i)a_{ik}n_i \tag{67}$$

which equals

$$\left(\sum a_{ik}\nabla_{ik}\right)\left(\sum f_i n_i\right) = v(f). \tag{68}$$

Consequence 3.34. Our geometric interpretation of the algebraic objects reproduces the relation:

$$\langle df|v\rangle = v(f) \tag{69}$$

known to hold in ordinary differential geometry, as is the case for the following relations, and shows that the definitions made above seem to be natural. Furthermore, vector and covector fields are left modules under the action of $\mathcal{A}$:

$$\left(\sum f_i n_i\right)\left(\sum a_{ik}\nabla_{ik}\right) := \sum f_i a_{ik}\nabla_{ik} \tag{70}$$

$$\left(\sum f_i n_i\right)\left(\sum g_{ik}d_{ik}\right) := \sum f_i g_{ik}d_{ik}. \tag{71}$$

Conclusion 3.35. The above shows that, in contrast to classical differential geometry, we have a dual pairing between vector and covector fields with the vector fields acting as 'twisted' derivations on the node functions while the corresponding differential forms obey the graded Leibniz rule like their classical counterparts.

Another important geometrical concept is the notion of '*connection*' or '*covariant derivative*'. Starting from the abstract concept of (linear) connection in the sense of Koszul it is relatively straightforward to extend this concept to the noncommutative situation, given a '*finite projective module*' over some algebra $\mathcal{A}$ (instead of the sections of a vector bundle over some manifold $\mathcal{M}$, the role of $\mathcal{A}$ being played by the functions over $\mathcal{M}$; see e.g. [23, 25]. As to various refinements and improvements cf e.g. [31] and further references given there).

Without going into any details we want to briefly sketch how the concept of connection can be immediately implemented in our particular model without referring to the more abstract work. We regard (in a first step) a connection as a (linear) map from the fields of tangent vectors to the tensor product of tangent vectors and dual differential forms as defined above (and having certain properties).

Definition/observation 3.36 (connection). A field of connections, Γ, is defined at each node n_i by a (linear) map:

$$\nabla_{ik} \to \gamma^j_{kl}(n_i) \cdot \nabla_{ij} \otimes d^l_i \tag{72}$$

where the index i plays rather the role of the 'coordinate' n_i, the index l is raised in order to comply with the summation convention. The γ^j_{kl}'s are called 'connection coefficients'. The corresponding 'covariant derivative' ∇ obeys the relations
(i)

$$\nabla(v + w) = \nabla(v) + \nabla(w) \tag{73}$$

(ii)

$$\nabla(f \cdot v) = v \otimes df + \nabla(v) \cdot f \tag{74}$$

(iii)

$$\nabla(\nabla_{ik}) = \Gamma(\nabla_{ik}) \qquad df = \sum (f_k - f_i) d_{ik}. \tag{75}$$

Remark. The tensor product in (ii) is understood as the pointwise product of fields at each node n_i, i.e. ∇_{ik} going with d_{ik}. This is to be contrasted with the abstract notion of tensor product in, e.g. the above differential algebra $\Omega(\mathcal{A})$ which does *not* act locally, the space consisting of, say, elements of the kind $n_1 \otimes n_2 \otimes \cdots \otimes n_k$. These different parallel structures over the same model shall be scrutinized in more detail elsewhere. Note that the above extra locality structure is a particular property of our model class and does not (openly) exist in the general approach employing arbitrary *projective modules* respectively *differential algebras*.

4. Intrinsic dimension in networks, graphs and other discrete systems

There exist a variety of concepts in modern mathematics which extend the ordinary or naive notion of *dimension* one is accustomed to in e.g. differential topology or linear algebra. In fact, *topological dimension* and related concepts are notions which are perhaps even closer to the underlying intuition (cf e.g. [32]).

Apart from the purely geometric concept there is also an important physical role being played by something like dimension, having e.g. pronounced effects on the behaviour of, say, many-body systems near their *phase transition points* or in the *critical region*.

But even in the case of, e.g. lattice systems they are usually treated as being embedded in an underlying continuous background space (typically Euclidean) which supplies the concept of ordinary dimension so that the *intrinsic dimension* of the discrete array itself does usually not openly enter the considerations.

Anyway, it is worthwhile even in this relatively transparent situation to have a closer look on the situations where attributes of something like dimension really enter the physical stage. Properties of models of, say, statistical mechanics are typically derived from the

structure of the microscopic interactions of their constituents. This then is more or less the only place where dimensional aspects enter the calculations.

Naive reasoning might suggest that it is something like the number of nearest neighbours (in e.g. lattice systems) which encodes the dimension of the underlying space and influences via that way the dynamics of the system. However, this surmise, as we will show in the following, does not reflect the crucial point which is considerably more subtle.

This holds the more so for systems which cannot be considered as being embedded in a smooth regular background and hence do not inherit their dimension from the embedding space. A case in point is our primordial network in which Planck-scale physics is assumed to take place. In our approach it is in fact exactly the other way round. Smooth spacetime is assumed to emerge via a *phase transition* or a certain *cooperative behaviour* and after some '*coarse graining*' from this more fundamental structure.

Problem 4.1. Formulate an intrinsic notion of dimension for model theories without making recourse to the dimension of some continuous embedding space.

As a first step we will show that graphs and networks as introduced in the preceding sections have a natural metric structure. We have already introduced a certain neighbourhood structure in a graph with the help of the minimal number of consecutive bonds connecting two given nodes.

In a connected graph any two nodes can be connected by a sequence of bonds. Without loss of generality one can restrict oneself to *paths*. One can then define the length of a path (or sequence of bonds) by the number l of consecutive bonds making up the path.

Definition/observation 4.2. Among the paths connecting two arbitrary nodes there exists at least one with minimal length which we denote by $d(n_i, n_k)$. This d has the properties of a metric, i.e:

$$d(n_i, n_i) = 0 \tag{76}$$

$$d(n_i, n_k) = d(n_k, n_i) \tag{77}$$

$$d(n_i, n_l) \leqslant d(n_i, n_k) + d(n_k, n_l). \tag{78}$$

(The proof is more or less evident.)

Corollary 4.3. With the help of the metric one gets a natural neighbourhood structure around any given node, where $\mathcal{U}_m(n_i)$ comprises all the nodes, n_k, with $d(n_i, n_k) \leqslant m$, $\partial\mathcal{U}_m(n_i)$ the nodes with $d(n_i, n_k) = m$.

This natural neighbourhood structure enables us to develop the concept of an intrinsic dimension on graphs and networks. To this end one has at first to realize what property really matters physically (e.g. dynamically), independently of the kind of model or embedding space.

Observation 4.4. The crucial and characteristic property of, say, a graph or network which may be associated with something like dimension is the number of 'new nodes' in $\mathcal{U}_{m+1}$ compared to $\mathcal{U}_m$ for m sufficiently large or $m \to \infty$. The deeper meaning of this quantity is that it measures the kind of 'wiring' or 'connectivity' in the network and is therefore a 'topological invariant'.

Remark. In the following I shall be very brief on this ramified topic as much more details have been presented and proved (in [8]). Instead I would like to discuss certain related ideas.

In many cases one expects the number of nodes in $\mathcal{U}_m$ to grow like some power D of m for increasing m. By the same token one expects the number of new nodes after an additional step to increase proportional to m^{D-1}. With $|\cdot|$ denoting the number of nodes we hence expect frequently the following to hold:

$$|\mathcal{U}_{m+1}| - |\mathcal{U}_m| = |\partial\mathcal{U}_{m+1}| = f(m) \tag{79}$$

with

$$f(m) \sim m^{D-1} \tag{80}$$

for large m.

Definition 4.5. 'The' intrinsic dimension D of a homogeneous (infinite) graph is given by

$$D - 1 := \lim_{m\to\infty} (\ln f(m)/\ln m) \tag{81}$$

or

$$D := \lim_{m\to\infty} (\ln |\mathcal{U}_m|/\ln m) \tag{82}$$

provided that a unique limit exists!

What does exist in any case is lim inf respectively lim sup which can then be considered as upper and lower dimension. If they coincide we are in the former situation. By 'homogeneous' we mean that the graph 'looks locally more or less the same' geometrically everywhere, in particular its dimension D should not depend on the reference point (and that the two definitions of graph dimension given above coincide!).

Remark.

(1) One might naively expect that '*regularity*', i.e. constant node degree, plus certain other conditions imply homogeneity but this is a highly nontrivial question. There are e.g. simple examples of regular graphs which do not 'look the same' around every node. That is, a so-called '*trivalent graph*', which is frequently taken as a discretized standard example in various versions of quantum gravity (see e.g. the above-mentioned literature about spin networks), is '*planar*', i.e. can be embedded in the plane. This may lead to the erroneous conclusion that its 'natural dimension' is two. On the other side, our analysis shows, that its 'intrinsic dimension' depends crucially on the details of its wiring and can even become infinite in case of a trivalent tree. Furthermore, it is a fascinating and nontrivial task to characterize, e.g. the dimension of regular lattices (or '*triangulations*') in a purely combinatorial manner without using a background space. This shall, however, be done elsewhere.

(2) On the other side we showed in [8] that, among other things, uniform boundedness of the node degree guarantees already the independence of D with respect to the reference point! Furthermore we were able to construct graph models for each given (fractal) dimension $D \in \mathbb{R}$. Due to discreteness, the relation between the two definitions of dimension may be a little subtle (as is the case for the various fractal dimensions; there may be exceptional graphs where they do not coincide).

(3) Other (but related) definitions of dimension are possible, incorporating e.g. the bonds instead of the nodes.

(4) For practical purposes one may also introduce a notion of local dimension around certain nodes or within certain regions of a regular graph if the above limit is approached sufficiently fast locally.

(5) Particularly interesting phenomena are expected to show up if this concept is applied within the regime of '*random graphs*' (see e.g. [11]).

(6) For some remarks concerning more or less related ideas in the literature see [8].

That this definition is reasonable can be seen by applying it to ordinary cases like regular translation invariant lattices. It is, however, not evident that such a definition makes sense for arbitrary graphs; in other words, a (unique) limit point may not always exist. It seems to be a highly nontrivial task to characterize the conditions which imply such a limit to exists.

Observation 4.6. For regular lattices D coincides with the dimension of the Euclidean embedding space D_E.

Proof. This can be rigorously proved but it is perhaps more instructive to simply draw a picture of the consecutive series of neighbourhoods of a fixed node for e.g. a two-dimensional Bravais lattice. It is fairly obvious that for m sufficiently large the number of nodes in $\mathcal{U}_m$ goes like a power of m with the exponent being the embedding dimension D_E as the Euclidean volume of $\mathcal{U}_m$ grows with the same power. □

Remark.

(1) For $\mathcal{U}_m$ too small the number of nodes may deviate from an exact power law which in general becomes only correct for sufficiently large m.

(2) The number of nearest neighbours, on the other side, does *not* influence the exponent but rather shows up in the prefactor. In other words, it influences $|\mathcal{U}_m|$ for m small but drops out asymptotically by taking the logarithm. For an ordinary Bravais lattice with N_C the number of nodes in a unit cell one has asymptotically

$$|\mathcal{U}_m| \sim N_C \cdot m^{D_E} \tag{83}$$

and hence

$$D = \lim_{m\to\infty} (\ln(N_C \cdot m^{D_E})/\ln m) = D_E + \lim_{m\to\infty} (N_C/\ln m) = D_E \tag{84}$$

independently of N_C.

Matters become much more interesting and subtle if one studies more general graphs than simple lattices. Note that there exists a general theorem (in fact a specialization of the more general result about simplicial complexes we mentioned in the preceding section to one-simplices), showing that practically every graph can be embedded in $\mathbb{R}^3$ and still quite a few in $\mathbb{R}^2$ (*planar graphs*).

This shows again that something like the dimension of the embedding space is in general not a characteristic property of a network or graph. On the contrary, it is generically entirely uncorrelated with its *intrinsic dimension* we defined above. An extreme example is a '*tree graph*', i.e. a graph without '*loops*'. In the following we study an infinite, regular tree graph with node degree 3, i.e. three bonds branching off each node. The absence of loops means that the '*connectivity*' is extremely low which results in an exceptionally high '*dimension*' as we will see.

Starting from an arbitrary node we can construct the neighbourhoods $\mathcal{U}_m$ and count the number of nodes in $\mathcal{U}_m$ or $\partial\mathcal{U}_m$. $\mathcal{U}_1$ contains three nodes which are linked with the reference node n_0. There are two other bonds branching off each of these nodes. Hence in $\partial\mathcal{U}_2 = \mathcal{U}_2 \backslash \mathcal{U}_1$ we have $3 \cdot 2$ nodes and by induction

$$|\partial\mathcal{U}_{m+1}| = 3 \cdot 2^m \tag{85}$$

which implies

$$D-1 := \lim_{m\to\infty}(\ln|\partial\mathcal{U}_{m+1}|/\ln m) = \lim_{m\to\infty}(m\cdot\ln 2/\ln m + 3/\ln m) = \infty. \tag{86}$$

Hence we have the following observation.

Observation 4.7 (regular tree). The intrinsic dimension of an infinite regular tree is ∞ and the number of new nodes grows exponentially like some $n(n-1)^m$ (n being the node degree).

Remark. $D=\infty$ is mainly a result of the absence of loops and of regularity, in other words: there is exactly one path, connecting any two nodes. This is usually not so in other graphs, e.g. lattices, where the number of new nodes grows at a much slower pace (whereas the number of nearest neighbours can nevertheless be large). This is due to the existence of many loops such that many of the nodes which can be reached from, say, a node of $\partial\mathcal{U}_m$ by one step are already contained in $\mathcal{U}_m$ itself. On the other hand we recently constructed [8] tree graphs having an arbitrary fractal dimension (they are, however, not regular, i.e. having a varying node degree, in particular they have a lot of endpoints). Hence the absence of loops is not the only reason for large graph dimensions.

We have seen that for, say, lattices the number of new nodes grows like some fixed power of m while for, say, regular trees m occurs in the exponent. The borderline can be found as follows.

Observation 4.8. If for $m\to\infty$ the average number of nodes in $\mathcal{U}_{m+1}$ per node contained in $\mathcal{U}_m$ is uniformly away from zero or, stated differently

$$|\mathcal{U}_{m+1}|/|\mathcal{U}_m| \geqslant 1+\varepsilon \tag{87}$$

for some $\varepsilon\geqslant 0$ then we have exponential growth, in other words, the intrinsic dimension is ∞.

The corresponding result holds with $\mathcal{U}_m$ being replaced by $\partial\mathcal{U}_m$.

Proof. If the above estimate holds for all $m\geqslant m_0$ we have by induction:

$$|\mathcal{U}_m| \geqslant |\mathcal{U}_{m_0}|\cdot(1+\varepsilon)^{m-m_0}. \tag{88}$$

□

Potential applications of this concept of intrinsic dimension are manifold. Our main goal is to develop a theory which explains how our classical spacetime and what we like to call the '*physical vacuum*' has emerged from a more primordial and discrete background via some sort of phase transition (the first preliminary steps being done in [11]).

In this context we can also ask in what sense *macroscopic* spacetime dimension four is exceptional, i.e. whether it is merely an accident or whether there is a cogent reason for it. We hope that the above concept of intrinsic dimension together with the dynamics of its possible change may help to come to grips with such a fundamental question.

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